The Tensor Product as a Lattice of Regular Galois Connections*

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Abstract. Galois connections between concept lattices can be represented as binary relations on the context level, known as *dual bonds*. The latter also appear as the elements of the *tensor product* of concept lattices, but it is known that not all dual bonds between two lattices can be represented in this way. In this work, we define *regular* Galois connections as those that are represented by a dual bond in a tensor product, and characterize them in terms of lattice theory. Regular Galois connections turn out to be much more common than irregular ones, and we identify many cases in which no irregular ones can be found at all. To this end, we demonstrate that irregularity of Galois connections on sublattices can be lifted to superlattices, and observe close relationships to various notions of distributivity. This is achieved by combining methods from algebraic order theory and FCA with recent results on dual bonds. Disjunctions in formal contexts play a prominent role in the proofs and add a logical flavor to our considerations. Hence it is not surprising that our studies allow us to derive corollaries on the contextual representation of deductive systems.

1 Introduction

From a mathematical perspective, Formal Concept Analysis (FCA) [1] is usually considered as a formalism for syntactically representing complete lattices by means of formal contexts. A closer look reveals that this representation hinges upon the fact that the well-known (context) derivation operations of FCA constitute *Galois connections*[3] between certain power set lattices. Thus formal contexts can equally well be described as convenient representations of such Galois connections.

With this in mind, it should not come as a surprise that Galois connections between concept lattices have also been studied extensively. On the level of formal contexts, such Galois connections can be described through suitable types of binary relations, called *dual bonds* in the literature, which turn out to be a very versatile tool for further studies. Dual bonds arguably constitute a fundamental notion for the study of interrelations and mappings between concept lattices. Indeed, many well-known morphisms of FCA, such as *infomorphisms* and *scale measures*, have recently been recognized as special types of dual bonds [2]. We review some of the relevant results on dual bonds in Sect. 3.

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[3] In this work, we study Galois connections only in their classical *antitone* formulation, as is done in [1].

Now dual bonds themselves allow for a nice representation in terms of FCA: each extent of the *direct product* of two contexts is a dual bond between them. The concept lattice of the direct product is known as the *tensor product* of FCA. However, this representation of dual bonds is usually not complete: the majority of dual bonds, herein called *regular* dual bonds, appears in the direct product, but there can also be irregular ones for which this is not the case.

Interestingly, this situation is also reflected by the corresponding Galois connections and we can distinguish regular and irregular Galois connections on purely lattice theoretical grounds. This allows us to give a lattice theoretical characterization of the tensor product in Sect. 4. Due to this relation to the tensor product, there are always plenty of regular Galois connections while irregular ones can be very rare. A major goal of this work is to further explore this situation, thus shedding new light on the structure of Galois connections between concept lattices and on the lattice theoretical relevance of the tensor product. In particular, we identify various cases for which only regular Galois connections exist, such that the tensor product yields a complete representation of the according function space.

In Sect. 5, we observe close relationships to the notion of complete distributivity, which will be a recurrent theme in this work. Disjunctions in formal contexts play a prominent role in Sect. 6 and add a logical flavor to our considerations. Hence it is not surprising that our studies allow us to derive corollaries on the contextual representation of deductive systems. Moreover, in Sect. 7, we demonstrate that irregularity of Galois connections on sublattices can be lifted to superlattices, which allows us to establish further relationships with distributivity. Finally, Sect. 8 summarizes our results and points to various open questions that need to be addressed in future research.

2 Preliminaries and notation

Our notation basically follows [1], with a few exceptions to enhance readability for our purposes. Especially, we avoid the use of the symbol $'$ to denote the operations that are induced by a context. We shortly review the main terminology using our notation, but we assume that the reader is familiar with the notation and terminology from [1]. Our treatment also requires some familiarity with general notions from order theory [3].

A *(formal) context* $\mathbb{K}$ is a triple (G, M, I) where G is a set of *objects*, M is a set of *attributes*, and $I \subseteq G \times M$ is an *incidence relation*. Given $O \subseteq G$ and $A \subseteq M$, we define:

$$O^I := \{m \in M \mid g\ I\ m \text{ for all } g \in O\}, \qquad I(O) := \{m \in M \mid g\ I\ m \text{ for some } g \in O\},$$
$$A^I := \{g \in G \mid g\ I\ m \text{ for all } m \in A\}, \qquad I^{-1}(A) := \{g \in G \mid g\ I\ m \text{ for some } m \in A\}.$$

By $\mathsf{Ext}(\mathbb{K})$ and $\mathsf{Int}(\mathbb{K})$ we denote the lattices of extents and intents of $\mathbb{K}$, respectively, ordered by subset inclusion.

The complement of a context $\mathbb{K}$ is defined as $\mathbb{K}^{\mathsf{c}} = (G, M, \not I)$ with $\not I := (G \times M) \setminus I$. We remark that $\mathsf{Ext}(\mathbb{K}^{\mathsf{c}})$ in general has no simple relationship to $\mathsf{Ext}(\mathbb{K})$: even if two contexts represent isomorphic concept lattices, this is not necessarily true for their complements.

Finally, an antitone Galois connection $\phi = (\vec{\phi}, \overleftarrow{\phi})$ between complete lattices K and L is a pair of functions $\vec{\phi} : K \to L$ and $\overleftarrow{\phi} : L \to K$ such that $k \leq \overleftarrow{\phi}(l)$ iff $l \leq \vec{\phi}(k)$, for all $k \in K$, $l \in L$. Each component of a Galois connection uniquely determines the other, so we will often work with only one of the two functions. For further details, see [1].

3 Dual bonds and the tensor product

To represent Galois connections between concept lattices on the level of the respective contexts, one uses certain relations called *dual bonds*. In this section, we recount some results that are essential to our subsequent investigations. Details and further references can be found in [2].

Definition 1. *A* dual bond *between formal contexts* $\mathbb{K} := (G, M, I)$ *and* $\mathbb{L} := (H, N, J)$ *is a relation* $R \subseteq G \times H$ *for which the following hold:*

- *for every element* $g \in G$*,* g^R *(which is equal to* $R(g)$*) is an extent of* $\mathbb{L}$ *and*
- *for every element* $h \in H$*,* h^R *(which is equal to* $R^{-1}(h)$*) is an extent of* $\mathbb{K}$*.*

This definition is motivated by the following result.

Theorem 1 ([1, Theorem 53]). *Consider a dual bond R between contexts* $\mathbb{K}$ *and* $\mathbb{L}$ *as above. The mappings*

$$\vec{\phi}_R : \mathsf{Ext}(\mathbb{K}) \to \mathsf{Ext}(\mathbb{L}) : X \mapsto X^R \quad \text{and} \quad \overleftarrow{\phi}_R : \mathsf{Ext}(\mathbb{L}) \to \mathsf{Ext}(\mathbb{K}) : Y \mapsto Y^R$$

form an antitone Galois connection between the concept lattices of $\mathbb{K}$ *and* $\mathbb{L}$*.*

Given such a Galois connection $(\vec{\phi}, \overleftarrow{\phi})$*, the relation*

$$R_{(\vec{\phi},\overleftarrow{\phi})} := \left\{(g, h) \mid h \in \vec{\phi}(g^{II})\right\} = \left\{(g, h) \mid g \in \overleftarrow{\phi}(h^{JJ})\right\}$$

is a dual bond, and these constructions constitute a bijection between the dual bonds between $\mathbb{K}$ *and* $\mathbb{L}$*, and the Galois connections between* $\mathsf{Ext}(\mathbb{K})$ *and* $\mathsf{Ext}(\mathbb{L})$*.*

The previous result allows us to switch between different context representations of dual bonds in a canonical way. Indeed, consider contexts $\mathbb{K}$, $\mathbb{K}'$, $\mathbb{L}$, and $\mathbb{L}'$ such that $\mathsf{Ext}(\mathbb{K}) \cong \mathsf{Ext}(\mathbb{K}')$ and $\mathsf{Ext}(\mathbb{L}) \cong \mathsf{Ext}(\mathbb{L}')$. Then any dual bond R between $\mathbb{K}$ and $\mathbb{L}$ bijectively corresponds to a dual bond R' between $\mathbb{K}'$ and $\mathbb{L}'$ that represents the same Galois connection. We describe this situation by saying that R and R' are *equal up to isomorphism* (of concept lattices) or just *equivalent*.

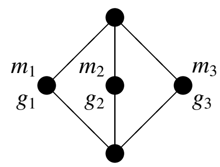
Since extents are closed under intersections, the same is true for the set of all dual bonds between two contexts. Thus the dual bonds (and hence the respective Galois connections) form a closure system, and one might ask for a way to cast this into a formal context which has dual bonds as concepts. An immediate candidate for this purpose is the direct product. Given contexts $\mathbb{K} := (G, M, I)$ and $\mathbb{L} := (H, N, J)$, the *direct product* of $\mathbb{K}$ and $\mathbb{L}$ is the context $\mathbb{K} \times \mathbb{L} := (G \times H, M \times N, \nabla)$, where ∇ is defined by setting $(g, h) \nabla (m, n)$ iff $g\ I\ m$ or $h\ J\ n$.

The *tensor product* of two complete lattices is defined as the concept lattice of the direct product of their canonical contexts. As shown in [1, Theorem 26], the tensor product does not depend on using canonical contexts: taking the direct product of any other contexts that represent the factor lattices yields an isomorphic result.

Proposition 1 ([4]). *The extents of a direct product* $\mathbb{K} \times \mathbb{L}$ *are dual bonds between the contexts* $\mathbb{K}$ *and* $\mathbb{L}$*.*

However, it is known that the converse of this result is false, i.e. there are dual bonds which are not extents of the direct product. This motivates the following definition.

$\mathbb{M}_3$	m_1	m_2	m_3
g_1	×		
g_2		×	
g_3			×



$\mathbb{N}_5$	m_1	m_2	m_3
g_1	×	×	
g_2		×	
g_3			×

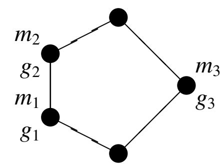


Fig. 1. The lattices M_3 and N_5 with their standard contexts.

Definition 2. *A dual bond R between $\mathbb{K}$ and $\mathbb{L}$ is* regular *if it is an extent of $\mathbb{K} \times \mathbb{L}$.*

Example 1. For some prototypical examples of irregular dual bonds, consider the formal contexts in Fig. 1. Observe that for any complete lattice L, the identity function is a Galois connection between L and its dual order L^{op}. For M_3, this identity is represented by the dual bond $\{(g_1, m_1), (g_2, m_2), (g_3, m_3)\}$ (between the standard context and its dual). For N_5 it is given as $\{(g_1, m_2), (g_1, m_1), (g_2, m_2), (g_3, m_3)\}$. Some easy computations show that both of these dual bonds are irregular.

The next section is concerned with showing that regularity of dual bonds is equivalent to suitable properties of the induced Galois connection. For regularity of dual bonds, the following characterization will be very helpful.

Theorem 2 ([2, Theorem 3]). *Consider contexts (G, M, I) and (H, N, J), and a dual bond $R \subseteq G \times H$. Then R is regular iff $R(g) = \bigcap_{m \in g^{I}} R(m^{I})^{JJ}$ for all $g \in G$.*

4 Regularity of Galois connections

The previous section suggests to extend the notion of regularity from dual bonds to their respective Galois connections.

Definition 3. *A Galois connection $\phi := (\vec{\phi}, \overleftarrow{\phi})$ between lattices K and L is* regular *if its associated dual bond R_ϕ between the canonical contexts of K and L is regular.*

We know that the lattice structure of the regular Galois connections coincides with the tensor product of the respective lattices, and that the latter does not depend on using canonical contexts in the definition. Whatever contexts are chosen for representing the given complete lattices, the structure of their regular dual bonds is always the same. This, however, does not say that they always represent the same set of Galois connections. In order to obtain this, one needs to show that the isomorphisms used in [1, Theorem 26] for showing the context independence of the tensor product preserve the represented Galois connections. However, using our prior insights on the connections between dual bonds, Galois connections, and the direct product, we can produce an alternative proof which is more suggestive in the current setting.

Lemma 1. *Consider a dual bond R between* $\mathbb{K}$ *and* $\mathbb{L}$*, and a subset A of the set of attributes of* $\mathbb{K}$*. We find that* $A^{R^\nabla} = \bigcap_{g \notin A^I} R(g)^J$*.*

Proof. We have $A^{R^\nabla} = \bigcap_{m \in A} R^\nabla(m)$ and, by [2, Lemma 3], this is equal to $\bigcap_{m \in A} R(m^{\not I})^J$. Since $\cdot^J$ transforms unions into intersections, the latter equals $\bigcap_{m\in A} \bigcap_{g \in m^{\not I}} R(g)^J$. In other words, the expression is the intersection of the intents $R(g)^J$ for all g such that $g \not I\, m$ for some $m \in A$. But this is just $\bigcap_{g \notin A^I} R(g)^J$ as required. □

The application of $\cdot^\nabla$ to a binary relation always yields a dual bond between the dual contexts, and thus a Galois connection between the dual concept lattices. Let us state the respective construction as a lattice theoretical operation on Galois connections.

Definition 4. *Consider a Galois connection* $\phi := (\vec{\phi}, \overleftarrow{\phi})$ *between complete lattices K and L. A pair of mappings* $\phi^\nabla := (\vec{\phi}^\nabla, \overleftarrow{\phi}^\nabla)$ *is defined as follows:*

$$\vec{\phi}^\nabla : K^{\mathsf{op}} \to L^{\mathsf{op}} : k \mapsto \bigvee \vec{\phi}(K \setminus \downarrow k) \qquad \textit{and} \qquad \overleftarrow{\phi}^\nabla : L^{\mathsf{op}} \to K^{\mathsf{op}} : l \mapsto \bigvee \overleftarrow{\phi}(L \setminus \downarrow l),$$

where $\cdot^{\mathsf{op}}$ *denotes order duals, and* $\bigvee$ *and* $\downarrow$ *refer to K and L, not to* K^{op} *and* L^{op}*.*

Lemma 2. *Consider a dual bond R between* $\mathbb{K}$ *and* $\mathbb{L}$*, and let* $\phi := (\vec{\phi}, \overleftarrow{\phi}) : \mathsf{Ext}(\mathbb{K}) \to \mathsf{Ext}(\mathbb{L})$ *be the according Galois connection as in Theorem 1.*

Then $\phi^\nabla = (\vec{\phi}^\nabla, \overleftarrow{\phi}^\nabla)$ *is a Galois connection from* $\mathsf{Ext}(\mathbb{K})^{\mathsf{op}}$ *to* $\mathsf{Ext}(\mathbb{L})^{\mathsf{op}}$*. Up to the isomorphism between the dual lattices of extents and the lattices of intents, it is the Galois connection associated with* R^∇*.*

Proof. Let $\psi = (\vec{\psi}, \overleftarrow{\psi})$ denote the Galois connection associated with R^∇. Given any intent A of $\mathbb{K}$, we compute

$$\vec{\psi}(A) = A^{R^\nabla} = \bigcap_{g \notin A^I} R(g)^J = \bigcap_{g^{II} \not\subseteq A^I} \vec{\phi}(g^{II})^J = \Big(\bigvee_{g^{II} \not\subseteq A^I} \vec{\phi}(g^{II}) \Big)^J$$

where we used Lemma 1 for the second equality, and where $\bigvee$ refers to the supremum of extents in $\mathsf{Ext}(\mathbb{L})$. Now it is easy to see that $\bigvee\{\vec{\phi}(g^{II}) \mid g^{II} \not\subseteq A^I\} = \bigvee\{\vec{\phi}(O) \mid O = O^{II},\ O \not\subseteq A^I\}$. Indeed, whenever $O = O^{II},\ O \not\subseteq A^I$ we find some $g \in O$ with $g \notin A^I$. But then $\vec{\phi}(O) \subseteq \vec{\phi}(g^{II})$ which allows for the desired conclusion. With this we conclude that $\vec{\psi}(A) = \big(\bigvee\{\vec{\phi}(O) \mid O = O^{II},\ O \not\subseteq A^I\}\big)^J = \big(\bigvee \vec{\phi}(\mathsf{Ext}(\mathbb{K}) \setminus \downarrow A^I)\big)^J$.

Now it is easy to see that $\vec{\psi}$ is just the composition of $\vec{\phi}^\nabla$ with the two lattice isomorphisms $\cdot^I : \mathsf{Int}(\mathbb{K}) \cong \mathsf{Ext}(\mathbb{K})^{\mathsf{op}}$ and $\cdot^J : \mathsf{Ext}(\mathbb{L})^{\mathsf{op}} \cong \mathsf{Int}(\mathbb{L})$. Since the property of being a Galois connection is invariant under isomorphism this establishes the claim. □

The previous result was stated for Galois connections between concept lattices only, which simplified the notation that was needed in the proof. Yet it is easy to see that the result extends to arbitrary Galois connections, since the claimed properties are invariant under isomorphism.

Also observe that the proof of Lemma 2 does not require the fact that ϕ is a Galois connection. This should not come as a surprise, since the operator $\cdot^\nabla$ on binary relations always produces an intent of the direct product, even if the input is not a dual bond.

Now we can easily derive the independence of regularity of Galois connections from the choice of canonical contexts for the representation via dual bonds.

Proposition 2. *Consider a dual bond R and a Galois connection ϕ that is, up to isomorphism of complete lattices, equal to the Galois connection induced by R. Then R is regular iff ϕ is regular. Moreover, for any Galois connection ψ, ψ^∇ is regular.*

Proof. R is regular iff $R = R^{\nabla\nabla}$. By Lemma 2, $\phi^{\nabla\nabla}$ is thus equivalent to the Galois connection induced by R and thus to ϕ. By similar reasoning, given that S is the canonical dual bond for ϕ, the Galois connection induced by $S^{\nabla\nabla}$ is equivalent to ϕ. But this implies that $S = S^{\nabla\nabla}$ such that ϕ is regular. The other direction is shown similarly.

This shows that regularity of Galois connections can be established by considering any (possibly non-canonical) representation by dual bonds. Hence, let R be any dual bond for ψ. Then, by Lemma 2, ϕ^∇ is represented (up to isomorphism) by R^∇ (as a dual bond between the dual contexts). Regularity of ϕ^∇ follows from regularity of R^∇. □

The above theorem asserts that regularity of dual bonds reflects a property not only of a particular dual bond, but of a whole class of dual bonds that represent the same Galois connection. This invariance under change of syntactic representation allows us to choose arbitrary contexts for studying regularity of Galois connections. As a corollary, we obtain that the structure of (binary) tensor products is independent of context representations as well. We find that the study of regularity of Galois connections or dual bonds is synonymous with the study of the structure of the tensor product.

In the remainder of this section, we provide some basic characterizations of regular Galois connections. Since the dual adjoints of a Galois connection uniquely determine each other, we state the following result only for one part of a Galois connection.

Theorem 3. *Given a mapping $\phi : K \to L$ between complete lattices K and L, the following are equivalent:*

(i) *ϕ is part of a regular Galois connection between K and L,*
(ii) $\phi = \phi^{\nabla\nabla}$,
(iii) *For all $k \in K$, $\phi(k) = \bigwedge_{m \not\geq k} \bigvee_{m \not\geq g} \phi(g)$.*

Proof. We first show that (i) is equivalent to (ii). By Proposition 2, every mapping of the form $\phi^{\nabla\nabla}$ is a part of a regular Galois connection. For the other direction, by Lemma 2, we find that $\phi^{\nabla\nabla}$ is equivalent to a part of the Galois connection associated with $R^{\nabla\nabla}$ where R is any dual bond for ϕ. The claim follows since $R^{\nabla\nabla} = R$ for regular Galois connections ϕ.

Equivalence of (ii) and (iii) is immediate by noting that $\phi^{\nabla\nabla}(k) = \bigwedge_{m \not\geq k} \bigvee_{m \not\geq g} \phi(g)$, where the second application of ∇ refers to the dual order, so that the order-related expressions have to be dualized. □

Note that this result establishes a purely lattice theoretical description of the tensor product of FCA, based on the closure operator (in the general order-theoretic sense) $\cdot^\nabla$ on Galois connections. An alternative characterization has been derived in [5], where a lattice-theoretical description of the original closure operator $\cdot^\nabla$ *on subsets of* $L \times K$ was given. The latter formulation is substantially more complex and hinges upon certain sets of filters, called *T-carpets*. The advantage of this approach is that it generalizes to n-ary direct products, while our description is specific to the binary case.

5 Regularity for completely distributive lattices

Though the above characterization of regularity is precise, its rather technical conditions are not fully satisfactory for understanding the notion. Especially, it does not significantly enhance our understanding of the sets of regular and irregular Galois connections as a whole. Next, we are going to explore several situations for which all Galois connections must be regular. In these cases, the representation of Galois connections through the direct product is exhaustive, and the tensor product is fully described as the *function space* of all Galois connections between two lattices.

Considering condition (iii) of Theorem 3, it should not come as a surprise that distributivity has an effect on regularity. The following characterization will be useful for formalizing this intuition.

Proposition 3. *A complete lattice K is completely distributive iff, for each pair of elements $g, m \in K$, if $m \not\geq g$, then there are elements $m', g' \in K$ such that:*

- $m' \not\geq g$ *and* $m \not\geq g'$,
- $K = \uparrow g' \cup \downarrow m'$.

Proof. For the proof, we use a result on completely distributive concept lattices. Clearly, K is completely distributive iff the concept lattice of its canonical context $\mathbb{K} = (K, K, \leq)$ is completely distributive. Now, according to [1, Theorem 40], the concept lattice of $\mathbb{K}$ is completely distributive iff for every pair $m \not\geq g$ there are elements m', g' such that $m' \not\geq g$, $m \not\geq g'$, and $g' \in k^{\leq\leq}$ for all $k \in K \setminus m'^{\leq}$. In the canonical context, the last condition is equivalent to saying that $g' \leq k$ for all $k \not\leq m'$. Thus $K = \uparrow g' \cup \downarrow m'$, as required. □

Note that the above also implies that $m \leq m'$ and $g \geq g'$. Now we can apply this result to establish the following sufficient condition for regularity of all Galois connections.

Theorem 4. *Consider complete lattices K and L. If either K or L is completely distributive, then all Galois connections from K to L are regular. Especially, this is the case if K or L are distributive and finite.*

Proof. Consider a Galois connection $\phi := (\vec{\phi}, \overleftarrow{\phi}) : K \to L$. Assume that L is completely distributive. For a contradiction, assume that ϕ is not regular. By Theorem 3, there is an element $k \in K$ such that $\vec{\phi}(k) \neq \bigwedge_{m \not\geq k} \bigvee_{m \not\geq g} \vec{\phi}(g)$. For notational convenience, we define $n := \vec{\phi}(k)$ and $h := \bigwedge_{m \not\geq k} \bigvee_{m \not\geq g} \vec{\phi}(g)$. It is easy to see that $n \leq h$, so we conclude that $n < h$ and thus $n \not\geq h$. This inequality satisfies the conditions of Proposition 3 and we obtain elements $h', n' \in L$ such that $n' \not\geq h$, $n \not\geq h'$, and $L = \uparrow h' \cup \downarrow n'$.

Now suppose that $\overleftarrow{\phi}(h') \geq k$. By anti-monotonicity of $\vec{\phi}$, this implies $\vec{\phi}(\overleftarrow{\phi}(h')) \leq \vec{\phi}(k) = n$. Since $h' \leq \vec{\phi}(\overleftarrow{\phi}(h'))$ (see, e.g., [1]), this entails $h' \leq n$, which contradicts the above assumptions on h'. Thus $\overleftarrow{\phi}(h') \not\geq k$, and we conclude that $h \leq \bigvee_{\overleftarrow{\phi}(h') \not\geq g} \vec{\phi}(g)$.

Now for any $g \in K$, if $\vec{\phi}(g) \geq h'$ then $\overleftarrow{\phi}(\vec{\phi}(g)) \leq \overleftarrow{\phi}(h')$ and thus $g \leq \overleftarrow{\phi}(h')$. Thus, whenever $\overleftarrow{\phi}(h') \not\geq g$, we find that $\vec{\phi}(g) \not\geq h'$. By our assumptions on h' and n', this shows that $\vec{\phi}(g) \leq n'$, and consequently $\bigvee_{\overleftarrow{\phi}(h') \not\geq g} \vec{\phi}(g) \leq n'$.

The conclusions of the previous two paragraphs imply that $h \leq n'$, which yields the desired contradiction. The claim for the case where K is completely distributive follows by symmetry. The rest of the statements is immediate since a finite lattice is distributive iff it is completely distributive. □

The above proof might seem surprisingly indirect, given that our first motivation for investigating distributivity possibly stems from the interleaved infimum and supremum operations of Theorem 3 (iii). Could a more direct proof just apply distributivity to exchange the position of these operations, thus enabling us to exploit the interplay of Galois connections and infima for further conclusions? The answer is a resounding "no": indeed, the infima and suprema from Theorem 3 distribute over each other in *any* finite lattice[4] but many finite lattices admit irregular dual bonds. Hence, just applying distributivity directly cannot suffice for a proof.

In the finite case, Theorem 4 shows that distributivity of lattices ensures that only regular Galois connections exist. In Sect. 7 we will see that distributivity is necessary as well.

6 Disjunctions in contexts

Further sufficient characterizations for regularity of dual bonds have been investigated in [2]. In this section, we combine these ideas with logical considerations along the lines of [7]. The results we obtain are specifically relevant for representations of logical and topological systems within FCA. The following properties of dual bonds constitute our starting point.

Definition 5. *Consider contexts $\mathbb{K} := (G, M, I)$ and $\mathbb{L} := (H, N, J)$. A relation $R \subseteq G \times H$ is (*extensionally*) *closed *if it* preserves *extents of $\mathbb{K}$, i.e. if for every extent O of $\mathbb{K}$ the image $R(O)$ is an extent of $\mathbb{L}$. R is (*extensionally*)* continuous *if its inverse is extensionally closed.*

In [2] continuity was used to establish relations to continuous functions between contexts, known as *scale measures*. Here, we are mostly interested in the following result, that is a corollary from [2, Theorem 4].

Proposition 4. *A dual bond R between $\mathbb{K}$ and $\mathbb{L}$ is regular, whenever it is closed as a relation from $\mathbb{K}^c$ to $\mathbb{L}$. By symmetry, the same conclusion follows if R is continuous as a relation from $\mathbb{K}$ to $\mathbb{L}^c$.*

Now continuity and closedness, while yielding sufficient conditions for a dual bond to be regular, are not particularly convenient as characterizations either. This is partially due to the fact that these properties, in contrast to regularity, are not independent from the contexts used in the presentation of a dual bond. Though this problem will generally persist throughout our subsequent considerations, the next lemma shows the way to a more convenient characterization.

[4] This is so because it holds for any continuous lattice since the sets $\{g' \mid m \ngeq g'\}$ are directed. See [6].

Lemma 3. *Consider a dual bond R between contexts $\mathbb{K} := (G, M, I)$ and $\mathbb{L} := (H, N, J)$, and an extent $O \in \mathsf{Ext}(\mathbb{K}^c)$. The following are equivalent:*

(i) *$R(O)$ is an extent of $\mathbb{L}$.*

(ii) *For every $h \in R(O)^{JJ}$, there is a set $X \subseteq O$ such that $h \in R(X)^{JJ}$ and there is $g \in O$ with $X \subseteq g^{\not{I}\not{I}}$.*

Proof. Clearly, if $R(O)$ is an extent, then for every $h \in R(O)^{JJ}$, there is an element $g \in O$ such that $h \in R(g)$. Since $g \in g^{\not{I}\not{I}}$, this shows that (i) implies (ii).

For the converse, let g be as in (ii). As observed in [2, Lemma 1], we find $g \in x^{II}$ for all $x \in X$. Given $x \in X$, $R^{-1}(y)$ is an extent of $\mathbb{K}$ for all $y \in R(x)$, such that we find $g \in R^{-1}(y)$ for all $y \in R(x)$, i.e. $R(x) \subseteq R(g)$. This yields $R(X) \subseteq R(g)$ which implies $h \in R(g)$ since the latter is an intent of $\mathbb{L}$. ◻

Intuitively, the previous lemma states that the closure of the image of a set O can be reduced to the closures of the images of single elements g, which are asserted by the definition of dual bonds. As shown in the proof, if the closure is given, then the existence of a sufficient amount of such elements g is certain and the subsets X of (ii) can be chosen as singleton sets. On the other hand, a sufficient condition for showing closure is obtained by requiring the existence of suitable g for arbitrary sets X. The *disjunctions* of X turn out to be just what is needed.

Definition 6. *Consider a context $\mathbb{K} := (G, M, I)$ and a set $X \subseteq G$. An object $g \in G$ is the* disjunction *of X, if, for any $m \in M$, we have that*

$$g\ I\ m \text{ iff there is some } x \in X \text{ such that } x\ I\ m.$$

Disjunctions in contexts introduce a logical flavor and have previously been studied in relation with the representation of deductive systems in FCA, e.g. in [8] or [7].

It is easy to see that X has a disjunction iff $X^{\not{I}\not{I}}$ is an object extent $g^{\not{I}\not{I}}$. A little reflection shows that the existence of disjunctions still strongly depends on the particular context used to represent some complete lattice. Intuitively, this is due to the fact that the concept lattice of $\mathbb{K}^c$ is not fully determined by the concept lattice of $\mathbb{K}$, but depends on the particular representation of $\mathbb{K}$.

Our strategy for deducing regularity of Galois connections from the above observations is as follows: given a Galois connection between complete lattices K and L, we try to find a corresponding dual bond R between contexts, such that the context for K has a "sufficient" amount of disjunctions to show closedness of R. Lemma 3 implies that the existence of arbitrary disjunctions certainly is sufficient in this sense; but weaker assumptions turn out to be sufficient in some cases. The existence of such a closed dual bond R then implies regularity of the Galois connection, even though both closedness and disjunctions may not be given for other representations of the same Galois connection. We discover that typical situations where many disjunctions exist again are closely related to distributivity.

Hence, our task is to seek context representations with a maximal amount of disjunctions. Unfortunately, the canonical context turns out to be mostly useless for this purpose. Indeed, given a complete lattice K, it is easy to see that $X \subseteq K$ has a disjunction in the canonical context iff X has a least element in K. In order to find contexts with more disjunctions, we state the following lemma.

Lemma 4. *Consider a complete lattice K and subsets J, $M \subseteq K$ such that M is $\bigwedge$-dense and J is $\bigvee$-dense in K. Then K is isomorphic to the concept lattice of $(J, M, \leq)$. Furthermore, a subset $X \subseteq J$ has a disjunction in $(J, M, \leq)$ iff*

(i) $\bigwedge X \in J$ *and*

(ii) $\uparrow \bigwedge X \cap M = \bigcup_{x \in X} \uparrow x \cap M$.

The disjunction then is given by the element $\bigwedge X$.

Proof. The claimed isomorphism of K and the concept lattice is a basic result of FCA, see [1]. According to the definition, g is a disjunction for X precisely when we find that for all $m \in M$, $g \leq m$ iff $x \leq m$ for some $x \in X$. This in turn is equivalent to $\uparrow g \cap M = \bigcup_{x \in X} \uparrow x \cap M$. Obviously this also implies that $g \leq \bigwedge X$. If $g < \bigwedge X$ then, by $\bigwedge$-density of M, there is some $m \in M$ with $m \geq g$ but $m \ngeq \bigwedge X$. But then $\uparrow g \cap M \subset \bigcup_{x \in X} \uparrow x \cap M$ and g cannot be the disjunction. This shows that $g = \bigwedge X$ is the only possible disjunction for X, and that this is the case iff (i) and (ii) are satisfied. □

This result guides our search for contexts with many disjunctions. Indeed, it is easy to see that for (i), it is desirable to have as many objects as possible, while (ii) is more likely to hold for a small set of attributes.

Note that Lemma 4 entails some notational inconveniences, since disjunctions are usually marked by the symbol $\bigvee$. Yet we obtain $\bigwedge$ since we work on disjunctions *of objects*. If one would prefer to do all calculations on attributes (which are often taken to represent formulae when modelling logical notions in FCA), one would obtain $\bigvee$ as expected.

In the finite case, finding a possibly small set of attributes for a given lattice is easy: the set of $\bigwedge$-irreducible elements is known to be the least $\bigwedge$-dense set. While this is also true for some infinite complete lattices, it is not required in this case.

Proposition 5. *Consider a complete lattice K such that the set of $\bigwedge$-irreducible elements $M(K)$ is $\bigwedge$-dense in K. For every set $X \subseteq K$, the following are equivalent:*

(i) X has a disjunction in the context $(K, M(K), \leq)$,

(ii) $k \vee \bigwedge X = \bigwedge_{x \in X}(k \vee x)$, for any $k \in K$.

In particular, the distributivity law $k \vee \bigwedge X = \bigwedge_{x \in X}(k \vee x)$ holds in K iff $(K, M(K), \leq)$ has all disjunctions.

Proof. For the forward implication, assume that (i) holds for some set $X \subseteq K$. For a contradiction, suppose that there is some $k \in K$ such that (ii) does not hold for k and X. Since $k \vee \bigwedge X \leq \bigwedge_{x \in X}(k \vee x)$ holds in any complete lattice, this says that $k \vee \bigwedge X < \bigwedge_{x \in X}(k \vee x)$. By $\bigwedge$-density of $M(K)$, there is some $m \in M(K)$ such that $k \vee \bigwedge X \leq m$ but $\bigwedge_{x \in X}(k \vee x) \nleq m$. The former shows that $k \leq m$ and $\bigwedge X \leq m$. Now by (i) and Lemma 4, $x \leq m$ for some $x \in X$, such that $(x \vee k) \leq m$. But then $\bigwedge_{x \in X}(k \vee x) \leq m$, which yields a contradiction.

For other direction, assume that (ii) holds for some set $X \subseteq K$. For a contradiction, suppose that (i) is not true for X. By Lemma 4, we find some attribute $m \in M(K)$ such that $\bigwedge X \leq m$ but $x \nleq m$ for all $x \in X$. We conclude that $m = m \vee \bigwedge X$. By (ii), this implies $m = \bigwedge_{x \in X}(m \vee x)$. Since m is $\bigwedge$-irreducible, there must be some $x \in X$ with $(m \vee x) = m$ and hence $x \leq m$; but this contradicts our assumptions. □

Note that most parts of the above proof do not make use of the $\bigwedge$-irreducibility of the chosen attribute-set. In particular, the implication from (i) to (ii) holds for arbitrary $\bigwedge$-dense sets of attributes. In the rest of the proof, irreducibility is used only in connection with the set X. Thus we obtain the following corollary.

Corollary 1. *Consider a complete lattice K such that the set of $\wedge$-irreducible elements $M_{Fin}(K)$ is $\bigwedge$-dense in K. Then K is distributive iff $(K, M_{Fin}(K), \leq)$ has all finite disjunctions.*

The previous result is useful for the subsequent consideration of logical deductive systems. Concerning regularity of dual bonds between finite lattices, we will establish a stronger characterization in Theorem 7 later on. We are now ready to combine the above results to derive further sufficient conditions for regularity of Galois connections.

Theorem 5. *Consider a complete lattice K where $M(K)$ is $\bigwedge$-dense and such that the distributivity law $k \vee \bigwedge X = \bigwedge_{x \in X}(k \vee x)$ holds for arbitrary $k \in K$, $X \subseteq K$. Then all Galois connections from K to any other complete lattice are regular.*

Proof. By Lemma 4, the concept lattice of $\mathbb{K} := (K, M(K), \leq)$ is isomorphic to K and, by Proposition 5, $\mathbb{K}$ has all disjunctions. Now any Galois connection from K to some complete lattice L corresponds to a dual bond R from $\mathbb{K}$ to the canonical context $\mathbb{L} := (L, L, \leq)$ of L. Now consider any extent O of $\mathbb{K}^c$. Obviously, the conditions of Lemma 3 (ii) are satisfied, where $X = O$ and g is the disjunction of O. Thus $R(O)$ is an extent of $\mathbb{L}$. Since O was arbitrarily chosen, this shows that R is closed from which we conclude that R is regular, as required. □

Complete lattices for which finite infima distribute over arbitrary suprema are also known as *locales*, and are the subject of study in point-free topology [9]. The reason is that the lattice of open sets of any topological space forms a locale. Thus the previous theorem can be considered as a statement about certain lattices of *closed* sets of topological spaces. On the other hand, the condition that $\bigwedge$-irreducibles be $\bigwedge$-dense is rather severe in this setting. Especialy, it would be possible that the conjunction of these assumptions already implies complete distributivity – see Sect. 8 for some discussions.

A statement similar to Theorem 5 can be made when restricting to finite disjunctions.

Theorem 6. *Consider a distributive complete lattice K where $M_{Fin}(K)$ is $\bigwedge$-dense. Then all Galois connections from K to any other complete algebraic lattice are regular. Especially, this applies to Galois connections from K to finite lattices.*

Proof. The proof proceeds as in Theorem 5. However, to apply Lemma 3, we note that for any $l \in R(O)^{\leq\leq}$, there is a *finite* set $Y \subseteq R(O)$ such that $l \in Y^{\leq\leq}$. This follows directly from algebraicity of L, see e.g. [1]. Thus there is a finite set $X \subseteq O$ with $Y \subseteq R(X)$ and its disjunction g allows us to invoke Lemma 3 as desired. □

Again this theorem can be related to topology, but in a way that is quite different from Theorem 5 and which finally relates disjunctions in contexts to logical disjunctions. The key observation is that $\bigwedge$-density of $\wedge$-irreducibles is the characteristic property for the *open* set lattices of certain topological spaces, called *sober* spaces in the literature [9]. Being locales as all topologies, these spaces are finitely distributive as well. Thus the conditions on K given in Theorem 6 are satisfied by the open set lattice of any sober space. These structures are commonly known as *spacial locales*, and have been studied extensively in research on point-free topology. We obtain the following corollary.

Corollary 2. *Every Galois connection between a spacial locale and an algebraic lattice is regular. In particular, every Galois connection between algebraic spacial locales is regular.*

To see how these observations are connected to logic, we have a brief look at the presentation of deductive systems in formal concept analysis. In general, deductive systems are characterized by a semantic *consequence relation* $\models$ between models and formulae of some logic. For an example, consider the consequence relation between models of propositional logic and propositional formulae.

Now such binary relations are naturally represented as formal contexts. This idea has, with more or less explicit reference to FCA, been investigated within *Institution Theory* [10] and the theory of *Information Flow* [8]. At this point, it is not apparent how this relates to topology, locales, and algebraicity. This relation can be established on quite general grounds, but here we just sketch the situation for propositional logic as an exemplary case.

Thus consider a language of propositional logic as a set of objects. For attributes, consider any set of models of some propositional logic theory.[5] With semantic consequence as incidence relation, this yields a "logical" context that represents a given theory. It is easy to see that disjunctions of the logic correspond to object-disjunctions within this context. On the other hand, disjunctions in the *complemented* context correspond to *conjunctions* of the logic. The according concept lattice is the lattice of logical theories over this background knowledge. In particular, object extents represent knowledge that is given by single formulae, while their order in the concept lattice describes entailment. As is well-known, the sublattice of object extents is a Boolean algebra in the propositional case. Furthermore, the lattice of propositional theories is known to be algebraic: every logical consequence can be derived from only a finite set of assumptions (in other words, propositional statements cannot describe infinite information).

Moreover, the complement of a logical context represents another interesting concept lattice: it is isomorphic to the open set lattice of a topological space, the so-called *Stone space* of the aforementioned Boolean algebra.[6] Basically, this is just an FCA version of Stone's famous representation theorem for Boolean algebras (see [3] for an introduction). It is well-known that open set lattices of Stone spaces are algebraic spacial locales, so that we can immediately conclude from Corollary 2 that every dual bond between complements of logical contexts in the above sense is regular. However, another consequence of our observations is more interesting from a logical perspective.

[5] This differs from the more common approach where formulae or "properties" are usually taken as attributes. This deviation ensures compatibility to our object-centered treatment. Also note that we do in general not consider the set of *all* propositional models, since the context would not contain much information in this case.

[6] Note that it is not the open set lattice, since the latter is not a closure system. However, the order-dual lattice of intents is exactly the according lattice of closed sets.

Corollary 3. *Consider contexts $\mathbb{K}$ and $\mathbb{L}$ that represent theories of propositional logic as described above. Then any dual bond from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}$ is regular. More specifically, it is closed from $\mathbb{K}$ to $\mathbb{L}$ and continuous from $\mathbb{K}^{\mathsf{c}}$ to $\mathbb{L}^{\mathsf{c}}$.*

Proof. Regularity follows immediately from Corollary 2 and the above remarks. For closedness, we can apply Lemma 3, using algebraicity of $\mathsf{Ext}(\mathbb{L})$ (the lattice of theories) and the availability of finite conjunctions in the propositional logic of $\mathbb{K}$. Likewise, for continuity, we combine algebraicity of $\mathsf{Ext}(\mathbb{K}^{\mathsf{c}})$ (the open set lattice) with finite propositional disjunctions of $\mathbb{L}$. □

The above formulation exhibits a seemingly strange twist in the dual bond, since we consider the complement of the context $\mathbb{K}$. However, this formulation fits well into our logical framework, since such dual bonds can be interpreted as proof theoretical *consequence relations* between two logical theories. To see this, note that a a logical implication $p \to q$ can be translated into $\neg p \vee q$. Based on this intuition, it makes perfect sense to interpret the dual bond of Corollary 3 as a set of logical implications. The defining conditions on dual bonds now state that the consequences of any single statement from $\mathbb{K}$ are deductively closed in $\mathbb{L}$, and that the sets of premises of a statement from $\mathbb{L}$ are deductively closed in $\mathbb{K}^{\mathsf{c}}$. Observe how this justifies regularity of all such consequence relations: given any binary relation between the logical languages, we can always derive an adequate consequence relation by computing missing deductive inferences. In logic, this process is usually described by application of certain deductive rules, while in FCA it corresponds to the concept closure within the direct product.

The reason for emphasizing closedness and continuity in Corollary 3 is that these properties enable us to compose consequence relations in a very intuitive way. Indeed, if p implies q, and q implies r, then one is usually tempted to derive that p implies r. Using dual bonds to represent implication, such reasoning is described by taking the relational product. Continuity and closedness ensure that this construction does again yield a dual bond as in Corollary 3. Hence we obtain a *category* of logical theories and consequence relations, the sets of morphisms of which can be described by the tensor product of FCA. However, to the authors' knowledge, the resulting categories have not yet been investigated with respect to their general properties or their relationship to other categories from algebra or order theory.

More details on deductive systems, consequence relations, and their contextual representation are given in [7]. In [11], consequence relations between separate logical theories (and languages) have been introduced proof-theoretically for positive logic (the logic of conjunctions and disjunctions), and the emerging categories were shown to be of topological and domain theoretical relevance. Much more general cases of *Stone duality* and their relation to FCA have been considered in [12].

7 Regularity for sublattices

In this section, we show that the irregularity of a Galois connection between sublattices can be lifted to a Galois connection between their respective superlattices, which enables us to improve our characterization of the interplay between distributivity and regularity.

Proposition 6. *Consider complete lattices K, L, U, and V such that $U \subseteq K$ and $V \subseteq L$ and*

(i) for any non-empty set $X \subseteq U$, we have $\bigwedge_U X = \bigwedge_K X$ and $\bigvee_U X = \bigvee_K X$,

(ii) for any non-empty set $Y \subseteq V$, we have $\bigwedge_V Y = \bigwedge_L Y$ and $\bigvee_V Y = \bigvee_L V$.

Then any irregular Galois connection between U and V induces an irregular Galois connection between K and L.

Proof. We use $\bot_K = \bigwedge K$ and $\top_K = \bigvee K$ to denote the least and greatest elements of K, respectively. Similar notations are used for U and L. Let $\phi := (\vec{\phi}, \overleftarrow{\phi}) : U \to V$ be an irregular Galois connection. Define a mapping $\psi : K \to L$ by setting

$$\psi(k) := \begin{cases} \top_L & \text{if } k = \bot_K \\ \vec{\phi}(\bigwedge\{u \in U \mid k \le u\}) & \text{if } \bot_K < k \le \top_U \\ \bot_L & \text{if } k \nleq \top_U \end{cases}$$

Note that we do not have to distinguish between infima in K and in U for the second case, since the considered set is always non-empty. We claim that ψ is one part of an irregular Galois connection from K to L.

Consider some set $X \subseteq K$. To see that ψ is a Galois connection, it suffices to show that $\psi(\bigvee X) = \bigwedge(\psi(X))$ (see [1, Proposition 7]). It $\bigvee X = \bot_K$ then X is either empty or contains only $\bot_K$. Both cases are easily seen to satisfy the claim. If $\bigvee X \nleq \top_U$, then there is some $x \in X$ such that $x \nleq \top_U$, i.e. $\psi(x) = \bot_L$. Again the claim is obvious.

It remains to consider the case where $\bot_K \le \bigvee X \le \top_U$. To this end, first note that $\bigwedge\{u \in U \mid x \le u \text{ for all } x \in X\} = \bigvee_{x \in X} \bigwedge\{u \in U \mid x \le u\}$ $(*)$. Indeed, the left hand side (lhs) is greater-or-equal than the right hand side (rhs). Assuming that it is strictly greater, the rhs is not among the u on the left, i.e. there is $x \in X$ with $x \nleq$ rhs. Since $x \le$ rhs $\le$ lhs, this yields a contradiction.

Furthermore, we can assume without loss of generality that $\bot_K \notin X$, since there is certainly some greater element in X as well, making $\bot_K$ redundant in all considered operations. We compute:

$$\begin{aligned} \psi(\bigvee X) &= \vec{\phi}(\bigwedge\{u \in U \mid \bigvee X \le u\}) \stackrel{*}{=} \vec{\phi}(\bigvee_{x \in X} \bigwedge\{u \in U \mid x \le u\}) \\ &= \bigwedge_{x \in X} \vec{\phi}(\bigwedge\{u \in U \mid x \le u\}) = \bigwedge_{x \in X} \psi(x). \end{aligned}$$

This finishes our proof that ψ is part of a Galois connection. To see that it is irregular, we use condition (iii) of Theorem 3. By the assumption that ψ is irregular, there is some $u \in U$ such that $\vec{\phi}(u) \neq \bigwedge_{m \in U,\, m \ngeq u} \bigvee_{g \in U,\, m \ngeq g} \vec{\phi}(g)$. Since the left hand side is always smaller-or-equal to the right hand side, this inequality is in fact strict. We have to show that $\psi(u) \neq \bigwedge_{m \ngeq u} \bigvee_{m \ngeq g} \psi(g)$. Since $\vec{\phi}(u) = \psi(u)$, this follows by showing that $\bigwedge_{m \in U,\, m \ngeq u} \bigvee_{g \in U,\, m \ngeq g} \vec{\phi}(g) \le \bigwedge_{m \ngeq u} \bigvee_{m \ngeq g} \psi(g)$. To obtain the latter, we observe that, for any $m \ngeq u$, there is some $n \in U$ with $n \ngeq u$ and $\bigvee_{g \in U,\, n \ngeq g} \vec{\phi}(g) \le \bigvee_{m \ngeq g} \psi(g)$. Indeed, consider $m \ngeq u$ and set $n := \bigvee\{u \in U \mid u \le m\}$. We claim that, for any $v \in U$ with $n \ngeq v$, $\vec{\phi}(v) \le \bigvee_{m \ngeq g} \psi(g)$. But this is obvious since $\vec{\phi}(v) = \psi(v)$ and $n \nleq v$ implies $m \nleq v$. By what was said before, this finishes the proof of irregularity of ψ. □

As a corollary to this result, we find that distributivity is necessary to assert that only regular Galois connections exist for some complete lattice.

Corollary 4. *If a complete lattice K has only regular Galois connections to any other lattice, then it is distributive.*

Proof. For a contradiction, assume that L is not distributive. Then L has either M_3 or N_5 as a sublattice. We have seen in Example 1 that both of these have an irregular Galois connection to some other lattice, so Proposition 6 yields the required contradiction. □

Summarizing our results, we obtain a satisfactory characterization of regularity for doubly-founded complete lattices.[7]

Theorem 7. *Given a doubly-founded complete lattice L, the following are equivalent:*

(i) L is distributive,

(ii) L has all disjunctions,

(iii) L has only regular Galois connections to any other complete lattice.

Proof. Recall that since L is doubly-founded, $M(L)$ is $\bigwedge$-dense in L, and that distributivity is equivalent to complete distributivity in this case [1, Theorem 41]. Thus (i) is equivalent to (ii) by Proposition 5. The implication from (i) to (iii) was stated in Theorem 4. The other direction follows from Corollary 4. □

8 Summary and outlook

In this work, we identified a novel property of Galois connections, dubbed regularity, which describes whether a Galois connection between two complete lattices is represented in their FCA tensor product. We characterized this property and identified several cases for which only regular Galois connections exist. These cases are of particular interest, since they enable us to represent the function space of all Galois connections by means of the tensor product, thus providing a lattice theoretical motivation for this construction.

Though we applied rather diverse proof strategies based on ideas from FCA, order algebra, and logic, many results expose relationships to notions of distributivity. It is known from Theorem 4 that complete distributivity of a lattice disallows irregular Galois connections to any other lattice, but a full characterization of this situation was only established for complete lattices that are doubly-founded (Theorem 7). We conjecture that a similar result holds for the general case, i.e. that a complete lattice admits only regular Galois connections to *any* other lattice iff it is completely distributive. Theorem 5 described other, seemingly weaker, conditions for enforcing regularity, but it is conceivable that these assumptions entail complete distributivity as well. Confirming or refuting these conjectures remains the subject of future work.

Apart form this immediate question, the present work shows many other directions for future research. First and foremost, we have concentrated on characterizations that refer to only one lattice at a time. This allowed us to identify specific situations where regularity is ubiquitious, but it also neglects the fact that in general both lattices contribute to regularity. Future investigations should take this into account, for example by

[7] Recall that every finite lattice is doubly-founded [1].

studying appropriate sublattices. Proposition 6 provides a theoretical foundation for this approach.

Considering mainly situations where no irregular Galois connections exist at all, we evaded the question for the role of irregular elements within the complete lattice of all Galois connections. Can the irregular elements be described lattice theoretically within this setting? We think that our results constitute the first steps towards such studies.

Last but not least, a completely different field of further questions was highlighted in Sect. 6, where we sketched fresh categories of deductive systems that use dual bonds as their morphisms. The study of these categories and their relevance in the field of logic/topology/domain theory remains open.

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